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AN ANALYSIS OF THE OPTIMIZATION OF A BEAM RIDER MISSILE SYSTEM

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SUMMARY

A transfer function is derived for a beam rider missile guidance system which is optimum when the target moves in a nonstationary way. The effects of acceleration limiting are considered and a discussion of the miss as a function of the various parameters which determine it is included. A form of design chart is presented which allows the immediate determination of the optimum under any set of target-missile conditions.

INTRODUCTION

With the recent progress in information theory and related techniques for optimization of systems operating in the presence of noise have come several applications of these optimization methods to the design of guided missile control systems (refs. 1 and 2). The results of these analyses have been transfer functions which specify the system which is optimum under the conditions assumed. The transfer functions thus obtained appear quite satisfactory, the mean-square miss distance associated with them being reasonably small and the functions themselves being not of an unrealistic form.

In order to arrive at their results, the authors of references 1 and 2 used the classical Wiener theory (ref. 3). As is well known, in order to apply the Wiener theory, it is necessary that the class of inputs to the desired system be stationary. Now, real targets may or may not maneuver in a nonstationary way. However this may be, one can easily find examples to which the Wiener theory as originally conceived does not apply: an example would be the case where the target maneuver consists of a step in acceleration. In such a case, if the Wiener theory alone were available as a tool to the designer, he probably would approximate the nonstationary maneuver by a stationary one. This implies that some improvement of such systems might be expected if more general target motions - ones involving no approximation - could be considered.

A method which allows the optimum to be determined when the inputs are not stationary was presented in references 4, 5, and 6. It is the purpose of this report to apply this method to the optimization of a beam rider control system. Our purpose here is exemplary; although an optimum transfer function is presented, we shall be more interested here in indicating how the method can be applied to missile problems than in specifying an optimum one. Thus, the point of view of the preceding paragraph will not be adhered to strictly. Although certain assumptions will be made which simplify the work, the situations described here will be more realistic than those of references 1 and 2. It should be stressed, however, that the methods of references 4, 5, and 6 are sufficiently powerful that these assumptions can be eliminated.

It might be of interest, while on the subject of assumptions, to discuss one which is <u>not</u> made. In all the previous works on missile optimization, it was assumed that the target and the missile move in one plane. One may be certain, however, that if the pilot of a target bomber knows this, he will do his best to assure that he does not remain in the same plane as the missile. Consequently, we shall not make this assumption: the plane in which it will be assumed the target moves will have no invariant relation to the missile's initial position and the flight path of the missile will not be assumed to be planar at all.

The paper begins with a discussion of the optimum system and the corresponding minimum error. It is shown that this system can never actually achieve the indicated minimum miss, since in order to do so the level of acceleration required of the missile would be impossibly high. The effect of introducing a side condition that the root-mean-square acceleration shall not exceed a preassigned value is then considered. By a new method another system is derived; this system has the properties that its rms acceleration remains below the given value while the miss corresponding to it is, for a fairly wide range of limiting accelerations, only slightly larger than the minimum miss. A brief discussion of this miss as a function of the parameters which determine it is then given.

SYMBOLS

a	vertical acceleration of target, ft/sec^2
a_M	acceleration of missile, ft/sec^2
E	root-mean-square miss distance, ft
N	$N_x+N_y+N_z$
N_x,N_y,N_z	noise amplitudes (magnitudes of spectral densities at zero frequency) in the directions of the coordinate axes, ft^{2}sec
R	$\sqrt{x_o^2+y_o^2+z_o^2}$
t	time measured from instant of firing, sec

t_o time at which target begins maneuver, sec

t^* λt, dimensionless time

T time at which missile velocity is so reduced it can no longer capture target, sec

V horizontal speed of target, ft/sec

x_o, y_o, z_o initial position of target, ft

x_B, y_B, z_B present position of target, ft

x_M, z_M present coordinates of missile, ft

x_N, y_N, z_N error due to noise in measuring target's position, ft

λ $\left(\frac{\overline{a^2}}{NT}\right)^{1/6}$

μ $e^{i\pi/3}$

The letter φ with subscripts attached will be reserved for correlation functions. A bar over any quantity will denote its average value.

ASSUMPTIONS

Beam Rider

We shall assume first of all that the missile it is desired to design will be a beam rider. The source of the beam will be considered to be far away from the target, so that as the target maneuvers the beam moves only parallel to itself, without rotation.

Target Motion

It will be assumed here that the target is initially (i.e., at the time of firing of the missile) flying with constant speed along some straight line. At some time after firing, the target will maneuver. Due to the difficulty of maneuvering a large bomber and the short time of flight of most missiles, we shall assume that the pilot has time for but one maneuver before the attack situation is over; for definiteness at this time, we take this to be a pitching maneuver[1] in which the bomber either climbs or dives at some constant acceleration which may or may not be the maximum of which it is capable. We assume as an approximation that this vertical acceleration leaves the bomber's forward velocity unimpaired, so that the flight path is a parabola rather than a circle.

Select a coordinate system fixed in space with the X axis parallel to the target's initial velocity vector and the origin at the initial position of the missile. According to what has gone before, we have

$$x_B = x_o + Vt$$

$$y_B = y_o$$

$$z_B = \begin{cases} z_o & , \quad t < t_o \\ z_o + \frac{1}{2} a(t-t_o)^2 & , \quad t > t_o \end{cases}$$

where t_o denotes the time at which the bomber begins its maneuver.

The preceding equations represent a whole class of missile-bomber combinations. If all the parameters, x_o, V, a, etc., are given definite values, a particular attack is defined. At this point, we assume that we know approximately how far from the missile a target will usually be initially, how fast a modern bomber will be going, and how rapidly it can accelerate. More precisely, it will be assumed that probability distributions of the parameters x_o, y_o, z_o, V, and a are known. (As will be seen, the entire distributions will not actually be needed, since only the mean-square values of these parameters will occur in the work.)

With Av denoting the average with respect to all parameters, we can then compute the following correlation functions. (See ref. 4 for the complete definition of these functions.)

$$\begin{aligned} \varphi_{x_B x_B}(t,\tau) &= \mathrm{Av}[(x_o+Vt)(x_o+V\tau)] \\ &= \mathrm{Av}(x_o{}^2) + \mathrm{Av}(x_oV)(t+\tau) + \mathrm{Av}(V^2)t\tau \qquad (1) \end{aligned}$$

[1]It will be seen that the argument which follows remains unimpaired if the vertical plane in which we have assumed the target maneuvers is rotated. Therefore, the analysis which follows will apply equally well to situations where the bomber turns or turns and dives, etc.

If the initial position of the bomber is uncorrelated with its velocity, and if, with respect to the missile's initial position, the bomber is as likely to be going in one direction as another, we obtain $Av(x_oV) = 0$. Equation (1) then becomes

$$\varphi_{x_B x_B}(t,\tau) = \overline{x_o^2} + \overline{V^2}t\tau$$

Similarly,

$$\varphi_{y_B y_B}(t,\tau) = \overline{y_o^2}$$

To compute $\varphi_{z_B z_B}$, it is necessary to assume some probability distribution for the time t_o at which the target begins its maneuver. Although it is possible to make more realistic assumptions, here we shall insist that t_o is equally likely to have any value between 0 and T, the time at which the missile speed has fallen off so badly that it can no longer reach the target. This assumption is not too unrealistic since a pilot who is tense will usually maneuver when t_o is small; the more composed will wait longer.

Knowing these things, we can now write down $\varphi_{z_B z_B}$. Assume first that $0 \leq \tau \leq t$. Then,

$$\begin{aligned}\varphi_{z_B z_B}(t,\tau) &= Av[z_B(t)z_B(\tau)] \\ &= \frac{1}{T}\int_0^T Av^*[z_B(t)z_B(\tau)]dt_o \qquad (2)\end{aligned}$$

where Av^* denotes the average with respect to z_o and a alone. Equation (2) then gives

$$\begin{aligned}\varphi_{z_B z_B}(t,\tau) &= \frac{1}{T}\int_0^\tau Av^*\left\{\left[z_o + \frac{1}{2}a(t-t_o)^2\right]\left[z_o + \frac{1}{2}a(\tau-t_o)^2\right]\right\}dt_o + \\ &\quad \frac{1}{T}\int_\tau^t Av^*\left\{\left[z_o + \frac{1}{2}a(t-t_o)^2\right]z_o\right\}dt_o + \frac{1}{T}\int_t^T Av^*(z_o^2)dt_o \\ &= \overline{z_o^2} + \frac{\overline{a^2}}{120T}(10t^2\tau^3 - 5t\tau^4 + \tau^5), \qquad \text{for } 0 \leq \tau \leq t\end{aligned}$$

In deriving this result, we have, as before, assumed that $Av(z_O a) = 0$. It follows from equation (2) that $\varphi_{z_B z_B}(t,\tau) = \varphi_{z_B z_B}(\tau,t)$. Hence,

$$\varphi_{z_B z_B}(t,\tau) = \overline{z_o^2} + \frac{\overline{a^2}}{120T}\left(10t^3\tau^2 - 5t^4\tau + t^5\right), \qquad \text{for} \quad \tau > t$$

We define the correlation function of the bomber $\varphi_B(t,\tau)$ as the sum of the functions $\varphi_{x_B x_B}, \varphi_{y_B y_B}$, and $\varphi_{z_B z_B}$. Thus,

$$\varphi_B(t,\tau) = \overline{R^2} + \overline{V^2}t\tau + \frac{\overline{a^2}}{120T}\left(10t^2\tau^3 - 5t\tau^4 + \tau^5\right) \tag{3}$$

for $0 \leq \tau \leq t$.

It is important to notice that the correlation function (3) depends only on the mean-square values $\overline{R^2}$, $\overline{V^2}$, and $\overline{a^2}$. This means that if the plane in which the bomber's maneuver takes place is not vertical, the resulting correlation function will have the same form. Thus, by choosing $\overline{a^2}$ to be the mean-square bomber acceleration over all possible orientations of the plane in which he maneuvers, we shall be considering the more general problem of a bomber, flying initially on a straight line course, maneuvering in some unknown direction.

Noise

The measurements of target position will presumably be made by radar or some other device which will be subject to error. It will be assumed that the noise is uncorrelated with the target motion and is "white." Thus, the autocorrelation of the noise will be given in the x, y, and z directions, respectively, by

$$\varphi_{x_N x_N}(t,\tau) = N_x\delta(t-\tau)$$

$$\varphi_{y_N y_N}(t,\tau) = N_y\delta(t-\tau)$$

$$\varphi_{z_N z_N}(t,\tau) = N_z\delta(t-\tau)$$

where N_x, N_y, and N_z are constants, dependent upon the configuration of the target, and $\delta(t-\tau)$ is the Dirac δ function (ref. 7). We define the correlation function of the noise as the sum of these three functions:

$$\varphi_N(t,\tau) = N\delta(t-\tau) \tag{4}$$

where $N = N_x + N_y + N_z$.

Inputs

The inputs to the missile consist of the sum of the target motion and the noise in each direction. Now, in general, the attack may take place from any direction relative to the target, even though certain directions may be more probable than others. However, the radar only perceives motions as if they occurred in a plane perpendicular to the beam. Thus, the information sent the missile is not actual target motion, but the projection of this motion onto such a plane. In an attack from the beam with the missile and the target in the same horizontal plane, these two motions coincide. In other situations, the effect of the deviation from a beam attack is the apparent reduction of the target velocity and acceleration. Thus, we may assume a beam attack by introducing the notion of an apparent target whose velocity and acceleration are less than that of the actual target.

Looked at another way, what we are saying is that the procedure of optimizing a beam rider may take place as if the plane in which the target is moving were orthogonal to the beam. In the final answer, the values of $\overline{V^2}$ and $\overline{a^2}$ which should be used are the apparent values averaged over all possible attack directions rather than the true values.

OPTIMIZATION

The Minimization Criterion

The over-all missile-beam system will generally be described by two transfer functions, one for the horizontal and one for the vertical direction. Although it is theoretically possible to attain a smaller mean-square error by actually using two different transfer functions, in this report we shall consider only cruciform missiles, so that the transfer function is the same in both directions.

According to the discussion in the preceding section, we may assume a beam attack, so that the y axis is parallel to the beam. Let $g(t,\tau)$ denote the impulse response[2] of the missile-beam system. Then, the output

[2]This response depends upon two variables since the system may, in general, be time-varying.

of the missile will be given by

$$x_M(t) = \int_0^t g(t,\tau)[x_B(\tau) + x_N(\tau)]d\tau$$

$$z_M(t) = \int_0^t g(t,\tau)[z_B(\tau) + z_N(\tau)]d\tau$$

We desire to minimize the mean-square error

$$E^2 = Av[x_B(t) - x_M(t)]^2 + Av[z_B(t) - z_M(t)]^2$$

A little algebra yields

$$E^2 = \varphi_B(t,t) - 2\int_0^t g(t,\tau)\varphi_B(t,\tau)d\tau +$$

$$\int_0^t g(t,\tau)\int_0^t g(t,\sigma)\varphi_B(\tau,\sigma)d\sigma\ d\tau + N\int_0^t g^2(t,\tau)d\tau\ , \qquad t \geq 0 \quad (5)$$

where equation (4) has been used to give an expression for φ_N. This is a minimum (cf. refs. 5 and 8) if and only if

$$\varphi_B(t,\tau) = \int_0^t g(t,\sigma)\varphi_B(\tau,\sigma)d\sigma + Ng(t,\tau)\ , \quad \text{for} \quad 0 \leq \tau \leq t \qquad (6)$$

where φ_B is given by equation (3).

Equation (6) is an integral equation which must be solved for the optimum impulse response $g(t,\tau)$.

The Optimum

Although it is a lengthy procedure, the method of reference 4 (or, more easily that of ref. 5) can be used to solve equation (6). The optimum impulse response one obtains is too complicated for there to be much advantage in writing it down, especially since a slight approximation simplifies it enormously. This approximation will be discussed in the next section. One can say, however, that everything turns out to be a function of the dimensionless time

$$t^* = \lambda t \tag{7}$$

where

$$\lambda = \left(\frac{\overline{a^2}}{NT}\right)^{1/6} \tag{8}$$

The optimum impulse response $g(t,\tau)$ can be used to compute the minimum rms error E_{min}. Again this function is rather complicated, but figure 1 can be used to find the minimum error as a function of time when the optimum missile is fired at a particular target. In order to interpret figure 1, we note that the minimum error has been broken up into four parts, a part E_R which represents error due to initial error at the time of firing, a part E_N due to noise, a part E_V due to target velocity, and a part E_B due to target maneuver. The total error is given by the square root of the sum of the squares of these four quantities.

As can be seen from the figure, E_R and E_V damp out rather rapidly. What remains can be approximated quite accurately by its value at infinity. One obtains from the figure that

$$\begin{aligned} E(\infty) &= \sqrt{E_N^2(\infty) + E_B^2(\infty)} \\ &= \sqrt{2N\lambda} \end{aligned} \tag{9}$$

This expression for the minimum error is very important and will be used frequently in the sequel.

AN APPROXIMATION

Equation (9) can be used as a check on the performance either of a given missile or of one designed by any heuristic process whatever. To see whether any missile transfer function, designed by whatever means, can be classed as "good" in the sense of this report, one has only to check on two points. First he has to see that the rms error, when $t = \infty$, is approached rapidly. This is always true of stable systems. Second, he may compare the error of his missile with equation (9) to see how far from the optimum he may be.

In what follows we shall make frequent use of the expressions as "t approaches infinity" and when "t is large." Actually, neither of these is quite accurate, since, of course, we shall never be concerned with times greater than T, when the attack is over. What really will be meant by large values of t are those values which, while less than T, are still far enough from zero that the expression $1 - e^{-\lambda t}$ may be approximated by unity. As can be seen from equation (8), λ is not usually very small and so "large values of t" may be quite close to zero.

In this section, we shall utilize the above approximation to reduce the formula for the optimum system to manageable proportions. Now, it happens that the optimum impulse response depends only on hyperbolic sines and cosines of t^* and $\tau^* = \lambda\tau$. Except near zero time, however, the following approximations are valid.

$$\cosh x = \sinh x = \frac{1}{2} e^{x}, \qquad x > 0$$

If these approximations are made in the expression for the optimum impulse response and if only fairly large values of t are considered, one obtains the simple approximation

$$g^*(t,\tau) = \lambda e^{-(t^*-\tau^*)} + \frac{1}{3}\lambda(1-\mu^2)e^{-\mu(t^*-\tau^*)} + \frac{1}{3}\lambda(1+\mu)e^{\mu^2(t^*-\tau^*)} \tag{10}$$

where λ is given by equation (8) and

$$\mu = e^{i\pi/3} = \frac{1}{2} + i\frac{\sqrt{3}}{2} \tag{11}$$

It should be noted that since g^* is a function of $t-\tau$ alone, it represents a time invariant system. Further, since the real parts of $-\mu$ and $+\mu^2$ are negative, the system is stable. The transfer function corresponding to the impulse response (10) is

$$\frac{1 + \frac{2}{\lambda} s + \frac{2}{\lambda^2} s^2}{\left(1 + \frac{1}{\lambda} s\right)\left(1 + \frac{1}{\lambda} s + \frac{1}{\lambda^2} s^2\right)} \tag{12}$$

Writing this in the form

$$\frac{1 + 2\frac{\zeta_n}{\omega_n} s + \frac{1}{\omega_n^2} s^2}{(1 + \tau s)\left(1 + 2\frac{\zeta_d}{\omega_d} s + \frac{1}{\omega_d^2} s^2\right)}$$

one obtains the values

$$\omega_n = \frac{\sqrt{2}}{2}\lambda \qquad \omega_d = \lambda$$

$$\zeta_n = \frac{\sqrt{2}}{2} \qquad \zeta_d = \frac{1}{2}$$

$$\tau = \frac{1}{\lambda}$$

where λ is given by equation (8).

The error corresponding to equation (10) can be computed by substituting (10) into (5). The exact procedure for doing this for any time invariant system will be indicated later on; it suffices here to say that the error corresponding to $g(t,\tau)$ for large t is given again by formula (9). Thus, although the error of this approximate system may be larger than the minimum initially, this transient phase soon dies out and the error for large t is compromised not at all.

ACCELERATION DEMANDED OF THE MISSILE

We have thus far specified an impulse response (10), corresponding to the transfer function (12), which has an associated error (9). However, we have no guarantee that in order to achieve the performance indicated by (9) the missile may not be called upon to attempt impossible exertions. The rms acceleration of the missile, for example, may be required to be so great that the controls of the actual missile will be at the stops all of the time, thus making the linear analysis used here invalid. The system may, of course, be limited in other ways, but we shall consider only acceleration limiting as typical of the sort of situation which arises, and, in addition, as the most important type of limiting.

Now, whatever the inputs and whatever the impulse response $g(t,\tau)$ may be, the mean-square acceleration demanded of the missile is

$$\overline{a_M{}^2} = \mathrm{Av}\left\{\frac{d^2}{dt^2}\int_0^t g(t,\tau)\left[x_B(\tau) + x_N(\tau)\right]d\tau\right\}^2 +$$

$$\mathrm{Av}\left\{\frac{d^2}{dt^2}\int_0^t g(t,\tau)\left[z_B(\tau) + z_N(\tau)\right]d\tau\right\}^2 \tag{13}$$

Define

$$\left.\begin{aligned} x_I(t) &= x_B(t) + x_N(t) \\ z_I(t) &= z_B(t) + z_N(t) \end{aligned}\right\} \tag{14}$$

Substituting these definitions into (13) and differentiating, one obtains

$$\begin{aligned} \overline{a_M{}^2} \doteq\ & \mathrm{Av}\left\{g(t,t)\dot{x}_I(t) + [2g_t(t,t) + g_\tau(t,t)]x_I(t) + \int_0^t g_{tt}(t,\tau)x_I(\tau)\,d\tau\right\}^2 + \\ & \mathrm{Av}\left\{g(t,t)\dot{z}_I(t) + [2g_t(t,t) + g_\tau(t,t)]z_I(t) + \int_0^t g_{tt}(t,\tau)z_I(\tau)\,d\tau\right\}^2 \end{aligned}$$

where the subscripts on g denote partial differentiation.

Now let $\varphi_{\dot{x}_I\dot{x}_I}(t,\tau)$ denote the autocorrelation function of $\dot{x}_I$, $\varphi_{\dot{x}_I x_I}(t,\tau)$ the cross-correlation function of $\dot{x}_I$ with x_I, etc. Then, squaring and averaging, one obtains

$$\begin{aligned} \overline{a_M{}^2} =\ & g^2(t,t)\left[\varphi_{\dot{x}_I\dot{x}_I}(t,t)+\varphi_{\dot{z}_I\dot{z}_I}(t,t)\right]+2g(t,t)\left[2g_t(t,t)+g_\tau(t,t)\right]\left[\varphi_{\dot{x}_I x_I}(t,t)+\varphi_{\dot{z}_I z_I}(t,t)\right] + \\ & \left[2g_t(t,t)+g_\tau(t,t)\right]^2\left[\varphi_{x_I x_I}(t,t)+\varphi_{z_I z_I}(t,t)\right]+2\int_0^t g(t,\tau)\left\{g(t,t)\left[\varphi_{\dot{x}_I x_I}(t,\tau)+\varphi_{\dot{z}_I z_I}(t,\tau)\right] + \right. \\ & \left.\left[2g_t(t,t)+g_\tau(t,t)\right]\left[\varphi_{x_I x_I}(t,\tau)+\varphi_{z_I z_I}(t,\tau)\right]\right\}d\tau+\int_0^t g_{tt}(t,\tau)\int_0^t g_{tt}(t,\sigma)\left[\varphi_{x_I x_I}(\tau,\sigma)+\varphi_{z_I z_I}(\tau,\sigma)\right]d\sigma\,d\tau \end{aligned} \tag{15}$$

Consider now the first bracketed term in this equation. It contains, according to equations (14), a term involving the noise. This term is

$$\varphi_{\dot{x}_N\dot{x}_N}(t,t) + \varphi_{\dot{z}_N\dot{z}_N}(t,t) = \mathrm{Av}[\dot{x}_N(t)\dot{x}_N(\tau) + \dot{z}_N(t)\dot{z}_N(\tau)]\Big|_{\tau=t}$$

$$= \frac{\partial^2}{\partial t\,\partial\tau}\mathrm{Av}[x_N(t)x_N(\tau) + z_N(t)z_N(\tau)]\Big|_{\tau=t}$$

$$= \frac{\partial^2}{\partial t\,\partial\tau}\varphi_N(t,\tau)\Big|_{\tau=t} \tag{16}$$

Now, if the noise is white, φ_N is given by equation (4) and so (16), which is to be evaluated at $\tau = t$, is infinite. Even if the strictly unrealizable assumption that the noise is white is not made, it can be expected that this term will be unduly large. In order to eliminate it then, we set, according to equation (15),

$$g(t,t) = 0 \tag{17}$$

Similarly, the third bracketed term in (15) gives

$$2g_t(t,t) + g_\tau(t,t) = 0 \tag{18}$$

We now specify (17) and (18) as inviolable conditions which any satisfactory missile must satisfy in order that the missile acceleration which is called for be not too large. In this case, (15) gives for the acceleration of the missile

$$\overline{a_M^2} = \int_0^t g_{tt}(t,\tau)\int_0^t g_{tt}(t,\sigma)\left[\varphi_{x_Ix_I}(\tau,\sigma) + \varphi_{z_Iz_I}(\tau,\sigma)\right]d\sigma\,d\tau \tag{19}$$

Let us see if the missile defined by the impulse response g^* of equation (10) satisfies conditions (17) and (18). We have from (10) that

$$g^*(t,t) = \lambda\left(1 + \frac{1-\mu^2}{3} + \frac{1+\mu}{3}\right)$$

$$= \frac{\lambda}{3}(5 + \mu - \mu^2)$$

$$= 2\lambda$$

since $\mu = e^{i\pi/3}$. The constant $\lambda \neq 0$, meaning that condition (17) is not satisfied. Thus, we may conclude that the acceleration demanded of the system described by (10) will be so large that the system is unsatisfactory.

OPTIMIZATION IN THE PRESENCE OF CONDITIONS (17) AND (18)

We have seen that the system which has been derived turns out to be unsatisfactory from the point of view of acceleration limiting. It appears that we might proceed using the ideas of reference 9, minimizing not the error E^2, but the quantity

$$E^2 + \gamma \overline{a_M^2} \tag{20}$$

where γ is a constant. The idea here is that when γ is large, effectively we would be minimizing $\overline{a_M^2}$ while, when γ is small, E^2 would be minimized. Consequently, the thought is that there might be a value of γ which while keeping E^2 near the minimum value (9) still does not allow $\overline{a_M^2}$ to get too large. However, in our case, the quantity (20) has no minimum, as can be shown by a straightforward (though lengthy) computation. Hence, we must proceed otherwise.

What we shall do is utilize a hint given us by the form of $g(t,\tau)$ and actually compute the error using formula (5). It will turn out that conditions necessary in order that E^2 be small will then become evident by inspection.

The hint is that $g(t,\tau)$ represents a time-invariant system. Thus, if we decide to look only at time-invariant systems, we may expect, in view of the fact that (12) is time-invariant, to be able to come upon a fairly good result.

Optimization

Now, the most general time-invariant system has an impulse response which is a sum of exponentials, provided only that the roots of the characteristic equation are distinct.[3] Hence, we set

$$g(t,\tau) = g(t-\tau) = \Sigma a_n e^{-\lambda_n (t-\tau)} \tag{21}$$

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[3]Even if they are not, the form of the impulse response is a limit of a sum of exponentials as the roots become coincident.

We may now substitute (21) into the expression (5) for the error. This procedure is simplified by breaking the error up into four parts, as in figure 1. Thus, we define, according to the expressions (3) and (4) for the correlation functions φ_B and φ_N,

$$\left.\begin{aligned}
E_R^2(t) &\equiv \overline{R^2}\left[1 - 2\int_0^t g(t,\tau)d\tau + \int_0^t g(t,\tau)\int_0^t g(t,\sigma)d\sigma\ d\tau\right] \\
&= \overline{R^2}\left[1 - \int_0^t g(t,\tau)d\tau\right]^2 \\
E_V^2(t) &\equiv \overline{V^2}\left[t^2 - 2t\int_0^t \tau g(t,\tau)d\tau + \int_0^t g(t,\tau)\int_0^t g(t,\sigma)\tau\sigma\ d\sigma\ d\tau\right] \\
&= \overline{V^2}\left[t - \int_0^t \tau g(t,\tau)d\tau\right]^2 \\
E_N^2(t) &\equiv N\int_0^t g^2(t,\tau)d\tau \\
E_B^2(t) &\equiv \frac{\overline{a^2}}{120T}\left[6t^5 - 2\int_0^t g(t,\tau)(10t^2\tau^3 - 5t\tau^4 + \tau^5)d\tau\ + \right. \\
&\quad \int_0^t g(t,\tau)\int_0^\tau g(t,\sigma)(10\tau^2\sigma^3 - 5\tau\sigma^4 + \sigma^5)d\sigma\ d\tau\ + \\
&\quad \left.\int_0^t g(t,\tau)\int_\tau^t g(t,\sigma)(10\tau^3\sigma^2 - 5\tau^4\sigma + \tau^5)d\sigma\ d\tau\right]
\end{aligned}\right\} \quad (22)$$

In equation (21), $g(t,\tau)$ is a function $g(t-\tau)$ of the difference of t and τ. This means that formulas (22) can be written

$$E_R^2(t) = \overline{R^2}\left[1 - \int_0^t g(\tau)d\tau\right]^2$$

$$E_V^2(t) = \overline{V^2}\left\{t\left[1 - \int_0^t g(\tau)d\tau\right] + \int_0^t \tau g(\tau)d\tau\right\}^2$$

$$E_N^2(t) = N\int_0^t g^2(\tau)d\tau$$

$$E_B^2(t) = \frac{\overline{a^2}}{120T}\left\{6t^5 - 2\int_0^t g(\tau)[(t-\tau)^5 - 5t(t-\tau)^4 + 10t^2(t-\tau)^3]d\tau + \right.$$

$$\left. 2\int_0^t g(\tau)\int_0^\tau g(\sigma)[(t-\tau)^5 - 5(t-\tau)^4(t-\sigma) + 10(t-\tau)^3(t-\sigma)^2]d\sigma\ d\tau\right\}$$

Set

$$I_n(t) = \int_0^t \tau^n g(\tau)d\tau\ , \qquad n = 0,1, \ldots \tag{23}$$

Then

$$E_R^2(t) = \overline{R^2}\ [1 - I_o(t)]^2$$

$$E_V^2(t) = \overline{V^2}\left\{t[1 - I_o(t)] + I_1(t)\right\}^2$$

$$E_N^2(t) = N\int_0^t g^2(\tau)d\tau$$

$$E_B^2(t) = \frac{\overline{a^2}}{120T}\left\{6t^5[1-I_o(t)]^2+30t^4[1-I_o(t)]I_1(t) - \right.$$

$$20t^3\left[\left\{1-I_o(t)\right\} I_2(t)-2I_1^2(t)\right]-60t^2I_1(t)I_2(t)+30tI_2^2(t) +$$

$$\left. 2\int_0^t g(\tau)\left[\tau^5\left\{1-I_o(\tau)\right\}+5\tau^4I_1(\tau)-10\tau^3I_2(\tau)\right]d\tau\right\}$$

Now, if we are to set $g(t)$ equal, as in equation (21), to a sum of exponentials, and if the system is to be stable, all of the quantities $I_n(t)$ will approach definite limits as $t \to \infty$. Consider, then, the t^5 term in the expression for $E_B{}^2$. Since $I_o(t)$ approaches a limit, this whole term will approach infinity as $t \to \infty$ unless $1-I_o(\infty) = 0$. This gives us one relation which $g(t)$ must satisfy:

$$I_o(\infty) = 1 \tag{24}$$

Similarly, the t^3 term in $E_B{}^2$ gives

$$I_1(\infty) = 0 \tag{25}$$

while the t term gives

$$I_2(\infty) = 0 \tag{26}$$

When these conditions are satisfied, we can say that

$$\left.\begin{aligned}
E_R{}^2(\infty) &= 0 \\
E_V{}^2(\infty) &= 0 \\
E_N{}^2(\infty) &= N\int_0^\infty g^2(\tau)d\tau \\
E_B{}^2(\infty) &= \frac{\overline{a^2}}{120T}\int_0^\infty g(\tau)\left\{\tau^5[1-I_o(\tau)] + 5\tau^4 I_1(\tau) - 10\tau^3 I_2(\tau)\right\} d\tau
\end{aligned}\right\} \tag{27}$$

The acceleration (19) of the missile can also be broken into four parts. Recalling that if $g(t,\tau) = g(t-\tau)$, then $g_{tt}(t,\tau) = \ddot{g}(t-\tau)$, we see that

$$\begin{aligned}
\overline{a_R{}^2} &\equiv \overline{R^2}\int_0^t \ddot{g}(t-\tau)\int_0^t \ddot{g}(t-\sigma)d\sigma\, d\tau \\
&= \overline{R^2}\left[\int_0^t \ddot{g}(\tau)d\tau\right]^2 \\
&= \overline{R^2}\left[\dot{g}(t) - \dot{g}(0)\right]^2 \\
&= \overline{R^2}\, \dot{g}^2(t)
\end{aligned} \tag{28}$$

by condition (18), which reads, in this time-invariant case,

$$\dot{g}(0) = 0 \tag{29}$$

Similarly, (17) becomes

$$g(0) = 0 \tag{30}$$

and so one can write

$$\overline{a_V{}^2} = \overline{V^2} g^2(t) \tag{31}$$

by using condition (30). Also,

$$\overline{a_N{}^2} = N\int_0^t \ddot{g}^2(\tau)\,d\tau \tag{32}$$

Now, set

$$\begin{aligned} J_n(t) &= \int_0^t \tau^n \ddot{g}(\tau)\,d\tau \\ &= t^n\dot{g}(t) - nt^{n-1}g(t) + n(n-1)I_{n-2}(t) \end{aligned} \tag{33}$$

Then, just as before,

$$\overline{a_B{}^2} \equiv \frac{\overline{a^2}}{120T}\int_0^t \ddot{g}(\tau)\int_0^\tau \ddot{g}(\sigma)[(t-\tau)^5 - 5(t-\tau)^4(t-\sigma) + 10(t-\tau)^3(t-\sigma)^2]\,d\sigma\,d\tau$$

$$= \frac{\overline{a^2}}{60T}\Big\{6t^5J_0{}^2(t) - 30t^4J_0(t)J_1(t) + 20t^3[J_0(t)J_2(t) + 2J_1{}^2(t)] -$$

$$60t^2J_1(t)J_2(t) + 30tJ_2{}^2(t) - 2\int_0^t \ddot{g}(\tau)[6\tau^5\dot{g}(\tau) - 15\tau^4 g(\tau) + 20\tau^3 I_0(\tau)]\,d\tau\Big\}$$

From (33), however, we see that if the system is stable (so that $g(\infty) = \dot{g}(\infty) = 0$), then $J_0(\infty) = J_1(\infty) = 0$. Also, using conditions (24), (25), and (26), we obtain $J_2(\infty) = 2$. Therefore, as $t \rightarrow \infty$,

$$\overline{a_B{}^2}(t) \sim \frac{\overline{a^2}}{60T}\Big\{60t - 2\int_0^t \ddot{g}(\tau)[6\tau^5\dot{g}(\tau) - 15\tau^4 g(\tau) + 20\tau^3 I_0(\tau)]\,d\tau\Big\} \tag{34}$$

Now, let $g(t)$ be given by (21). Conditions (24), (25), (26), (29), and (30) then become

$$\left.\begin{aligned} \sum \frac{a_n}{\lambda_n} &= 1 \\ \sum \frac{a_n}{\lambda_n^2} &= 0 \\ \sum \frac{a_n}{\lambda_n^3} &= 0 \\ \sum \lambda_n a_n &= 0 \\ \sum a_n &= 0 \end{aligned}\right\} \tag{35}$$

Notice that the first three of equations (35) are conditions for the error not to be too large, while the last two are conditions for the acceleration to be finite. Finally, we have

$$\begin{aligned} E^2(\infty) &= E_N^2(\infty) + E_B^2(\infty) \\ &= \sum_{m,n} \frac{a_m a_n}{\lambda_m + \lambda_n}\left(N + \frac{\overline{a^2}/T}{\lambda_m^3 \lambda_n^3}\right) \\ &= N \sum_{m,n} \frac{a_m a_n}{\lambda_m + \lambda_n}\left(1 + \frac{\lambda^6}{\lambda_m^3 \lambda_n^3}\right) \end{aligned} \tag{36}$$

where[4] λ is given by (8) and the sum runs over all m and n. Since we still are interested in reducing E^2 as far as possible, the problem now becomes one of minimizing (36) subject to the conditions (35).

Equation (21) can be substituted into (28), (31), (32), and (34) to yield an expression for the acceleration of the missile. Assuming stability, we see that $\overline{a_R^2}$ and $\overline{a_V^2}$ quickly approach zero. On the other hand,

$$\begin{aligned} \overline{a_M^2}(t) &\sim \overline{a_N^2} + \overline{a_B^2} \\ &= \overline{a^2}\,\frac{t}{T} + N \sum_{m,n} \frac{a_m a_n}{\lambda_m + \lambda_n}\left(\lambda_m^2 \lambda_n^2 + \frac{\lambda^6}{\lambda_m \lambda_n}\right) \end{aligned} \tag{37}$$

for large t.

[4]An alternative formulation of formulas (36) and (37) is given in the appendix.

It is very important to notice that there are infinitely many systems satisfying conditions (35). In fact, even if a characteristic equation (of sufficiently high order, see below) is given for the determination of the roots λ_n, there still exists a system (perhaps many such) satisfying (35). To see this, consider the λ_n's as fixed. Equations (35) then give relations among the a_n's. These, however, determine only the numerator of the transfer function. Hence, we can say that given a denominator one can find a corresponding numerator such that the system satisfies (35).

This result shows that not only can one use the ideas discussed here to design a missile but that one can very easily add a compensating network to an existing system so that the new over-all system will satisfy the conditions (35). This follows from the fact that equations (35) only give relations among the a_n's. Further, since the λ_n's are still arbitrary, they remain as parameters with respect to which the error (36) can be minimized.

The Transfer Function

Conditions (35) may easily be interpreted as conditions on the transfer function $G(s)$ corresponding to the impulse response $g(t)$. In fact, since

$$g(t) = \Sigma a_n e^{-\lambda_n t}$$

we have

$$G(s) = \sum \frac{a_n}{s+\lambda_n}$$

Consequently, the first three conditions (35) become

$$\left.\begin{aligned} G(0) &= 1 \\ G'(0) &= 0 \\ G''(0) &= 0 \end{aligned}\right\} \tag{38}$$

Consider next the last two of equations (35). They say that $g'(0) = g(0) = 0$. As is well known, however, this can only be so if the order of the denominator of g exceeds the order of the numerator by at least 3. Thus, we need only add the relation

$$(\text{order of denominator of } G) - (\text{order of numerator of } G) \geq 3 \tag{39}$$

to equations (38) to assure ourselves that all of equations (35) are satisfied.

Now, suppose

$$G(s) = \frac{P(s)}{Q(s)}$$

where P and Q are polynomials of order m and n, respectively. Equation (39) then reads

$$n - m \geq 3 \tag{40}$$

while equations (38) become

$$P(0) = Q(0)$$

$$P'(0) = Q'(0)$$

$$P''(0) = Q''(0)$$

Thus, equations (38) are satisfied if the first three terms (i.e., the constant, linear and quadratic terms) in P(s) and Q(s) are equal. We may, by dividing through by the constant, assume it to be unity and say that conditions (35) will be satisfied if G(s) has the form

$$G(s) = \frac{1+\alpha_1 s+\alpha_2 s^2+s^3 \sum_{\nu=0}^{m-3} \beta_\nu s^\nu}{1+\alpha_1 s+\alpha_2 s^2+s^3 \sum_{\nu=0}^{n-3} \alpha_{\nu+3} s^\nu} \tag{41}$$

and (40) holds.

Note that it follows from (40) and (41) that the order of the system is at least 5, since obviously, in view of (40), the lowest order system will occur if $m = 3$ and $\beta_0 = 0$ - in which case, again by (40), n must equal 5, at least.

SOME OPTIMUM TRANSFER FUNCTIONS

In this section, we wish actually to display some systems satisfying conditions (35) and minimizing (36), in order to show the effect on the missile acceleration.

As shown in the preceding section, any such system must be of the fifth order, at least. For simplicity, we shall only consider systems with this minimum order. Thus, according to equation (41) and the remarks following, we write

$$G(s) = \frac{1+\alpha_1 s+\alpha_2 s^2}{1+\alpha_1 s+\alpha_2 s^2+\alpha_3 s^3+\alpha_4 s^4+\alpha_5 s^5} \tag{42}$$

We know that the impulse response corresponding to (42) satisfies (35) if $\alpha_5 \neq 0$. Notice that, except for the further condition of stability, the five constants α are free and can be used to minimize (36). It is possible that with so much freedom, the error (36) - even when the constants a_n are subject to conditions (35) - can be reduced to the absolute minimum error (9). However, the minimization of (36) in this way is extremely complicated and the problem probably can only be solved by use of a method such as steepest descent (ref. 10) on a digital computer. Here, we shall solve a much less ambitious problem; however, we shall show that in order to reduce the missile acceleration very greatly, a large penalty in increased error need not be paid.

What we shall actually consider are special cases of (42) having the form

$$G(s) = \frac{1+2\left(1+\frac{1}{\beta}\right)\frac{s}{\lambda}+2\left(1+\frac{1}{\beta}\right)^2\frac{s^2}{\lambda^2}}{\left(1+\frac{s}{\lambda}\right)\left(1+\frac{s}{\lambda}+\frac{s^2}{\lambda^2}\right)\left(1+\frac{2s}{\beta\lambda}+\frac{2s^2}{\beta^2\lambda^2}\right)} \tag{43}$$

where, as before, λ is given by (8). The transfer functions (43) depend on a parameter β. Notice that as $\beta \to \infty$, (43) reduces to the optimum transfer function (12). Thus, for large β at least, one may expect the error corresponding to (43) to be somewhere in the neighborhood of the minimum error (9).

The actual error of the system (43) can be computed with the aid of (36). The computation, though long, is not impossible, and the final result is given in normalized form in figure 2. Note that as predicted, the curve is fairly flat, except at the very left-hand end.

In order to understand what significance the magnitude of β has, it is necessary to look at the mean-square acceleration of the missile. This was computed, using (37) with $t=T$ (the worst case). The result is shown in normalized form in figure 3 (the source of the parameter used as ordinate can be seen in the appendix). Figure 3 also contains the curve of figure 2.

Figure 3 represents the climax of all our efforts. In order to use the figure, one would have to decide what the rms acceleration capabilities of the missile were to be. Using this knowledge, one finds the appropriate $(\overline{a_M^2} - \overline{a^2})/N\lambda^5$ and reads off the corresponding value of abscissa β. This value of β is then used in equation (43) to give the optimum transfer function. The error corresponding to this system is then read at the same value of β. An example of this is given in a later section.

The Error

The error of the system (43) is plotted against the dimensionless missile acceleration parameter $A = (\overline{a_M^2} - \overline{a^2})/N\lambda^5$ in figure 4. Notice that the error E is determined by only two things, the minimum error E_{min} and this parameter A. An advantage of the present applications of the method of "optimization" is that the parameters of which E is a function are determined analytically.

Note further that over what might be considered some reasonable range of A - say, those values corresponding to the limits $4 \leq \sqrt{\overline{a_M^2}/\overline{a^2}} \leq 7$, a condition which with a "reasonable" choice of $\overline{a^2}$, N, and T is the range bounded by the dotted lines in figure 4 - the slope changes slowly. One of the principal advantages of the present outlook is that the curve of figure 4 gives the error over the practical range of A uniformly.

AN EXAMPLE

As an example of the use of figure 3, consider a target having the following characteristics[5]

$$\sqrt{\overline{a^2}} = 32.2 \text{ ft/sec}^2$$

$$N = 100 \text{ ft}^2\text{sec}$$

---

[5]The value of N given here is a reasonable one and is consistent with that used in earlier reports. In reference 1, for instance, the value of "noise magnitude" which was used was 15 ft^2/radian/sec; since our definition of N differs from that used in reference 1 by a factor of 2π we have set $N = 2\pi(15) \approx 100$.

Let us assume that the time of flight T is 10 seconds, and that the rms value of missile acceleration will be 7g's, so that

$$\sqrt{\overline{a_M^2}} = 225 \text{ ft/sec}^2$$

We then have

$$\lambda = \left(\frac{\overline{a^2}}{NT}\right)^{1/6}$$

$$= 1.00$$

Also,

$$\frac{\overline{a_M^2} - \overline{a^2}}{N\lambda^5} = 498$$

Hence, reading along the dotted lines in figure 3, we see that we must choose $\beta = 8.6$, which gives a value of the error of

$$E = 1.19 \; E_{min}$$

and, since

$$E_{min} = \sqrt{2N\lambda}$$

$$= 14.1 \text{ ft}$$

we have

$$E = 16.8 \text{ ft} \tag{44}$$

With $\beta = 8.6$ and $\lambda = 1.00$, the transfer function (43) becomes

$$G(s) = \frac{1+2.23s+2.49s^2}{(1+s)(1+s+s^2)(1+0.23s+0.027s^2)} \tag{45}$$

In the attack situation described, the system having the transfer function (45) will have the error (44).

CONCLUDING REMARKS

It has been shown how a beam rider missile system can be designed in an optimum way even when a target maneuver is a strictly nonstationary one. The maneuver was chosen here to be a step in target acceleration, the step being assumed to occur with equal likelihood anywhere in a finite interval of time. A simple formula was derived for the minimum error in these circumstances; this formula can be used as a criterion of merit against which the value of any beam rider, designed by whatever means, can be measured.

The optimum control system was shown to be unsatisfactory since the accelerations demanded of the missile were unduly large. Therefore, it was necessary to design it in a different way, taking into account the limited acceleration available to any real missile, and using the minimum error formula to decide how good the new system was. Two things resulted from this analysis. The first was the conclusion that the transfer function of any missile guidance system must be of a certain form (given by eqs. (40) and (41)) if either the missile acceleration or the miss distance is not to increase beyond all bounds.

The second result was presented in figure 3 and equation (43). Figure 3 may be used as a design chart to determine a satisfactory system transfer function in any given situation. Thus, having decided just how much acceleration the desired missile would be able to withstand, the designer can use figure 3 to determine a corresponding value of a certain parameter β. The transfer function is then determined by means of equation (43). The error of a missile system with this transfer function may then be read off figure 3.

Finally, the curve of error versus a dimensionless acceleration parameter - which is the only thing which determines the error - is given.

Ames Aeronautical Laboratory
National Advisory Committee for Aeronautics
Moffett Field, Calif., Sept. 30, 1957

APPENDIX

ALTERNATIVE FORMULATION OF EQUATIONS (36) AND (37)

In order to evaluate any system, it is necessary to calculate the error corresponding to it (of course) and its rms acceleration. With the proviso that conditions (29) and (30) hold, formulas (36) and (37) of the text can be used for this purpose. However, it is simpler computationally to consider a transformed version of these formulas. This version is described below.

Consider equation (36). We have

$$E^2(\infty) = N\sum_{m,n} \frac{a_m a_n}{\lambda_m+\lambda_n}\left(1 + \frac{\lambda^6}{\lambda_m{}^3\lambda_n{}^3}\right)$$

$$= N\sum_{m,n} a_m a_n \int_0^\infty \left(1 + \frac{\lambda^6}{\lambda_m{}^3\lambda_n{}^3}\right) e^{-(\lambda_m+\lambda_n)t} dt$$

$$= N\int_0^\infty \left[\left(\sum a_n e^{-\lambda_n t}\right)^2 + \lambda^6\left(\sum \frac{a_n}{\lambda_n{}^3} e^{-\lambda_n t}\right)^2\right] dt$$

and in view of equation (21),

$$= N\int_0^\infty [g^2(t) + \lambda^6 h^2(t)]dt$$

where

$$h(t) = \sum \frac{a_n}{\lambda_n{}^3} e^{-\lambda_n t}$$

Now, by a well-known formula for Laplace transforms (ref. 11)

$$\int_0^\infty g^2(t)dt = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} G(s)G(-s)ds$$

where $G(s)$ is the Laplace transform of $g(t)$. Also, again by (21),

$$h(t) = \frac{1}{2}\int_t^{\infty} (t-\tau)^2 g(\tau)d\tau$$

Consequently, the transform of $h(t)$ is

$$H(s) = \int_0^{\infty} g(t)\left[\frac{t^2}{2s} - \frac{t}{s^2} + \frac{1}{s^3} - \frac{e^{-st}}{s^3}\right] dt$$

$$= \frac{G''(0)}{2s} + \frac{G'(0)}{s^2} + \frac{G(0)}{s^3} - \frac{G(s)}{s^3}$$

However, as shown on page 20, conditions (35) entail (38), so that if conditions (35) are valid for the system under consideration,

$$H(s) = \frac{1-G(s)}{s^3}$$

Also,

$$\int_0^{\infty} h^2(t)dt = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} H(s)H(-s)ds$$

Therefore, finally,

$$E^2(\infty) = \frac{N}{2\pi i}\int_{-i\infty}^{i\infty}\left[G(s)G(-s)-\lambda^6 \frac{1-G(s)}{s^3}\,\frac{1-G(-s)}{s^3}\right] ds$$

$$= \frac{N}{2\pi i}\int_{-i\infty}^{i\infty} G(s)G(-s)ds - \frac{1}{2\pi i}\,\frac{\overline{a^2}}{T}\int_{-i\infty}^{i\infty} \frac{1-G(s)}{s^3}\,\frac{1-G(-s)}{s^3}\, ds \qquad \text{(A1)}$$

by virtue of the definition (8) of λ.

In a similar way, it can be shown that equation (37) is equivalent to

$$\overline{a_M^2}(T) = \overline{a^2} + \frac{N}{2\pi i}\int_{-i\infty}^{i\infty}\left[s^4G(s)G(-s) - \lambda^6\,\frac{1-G(s)}{s}\,\frac{1-G(-s)}{s}\right]ds \tag{A2}$$

$$= \overline{a^2}\left[1 - \frac{1}{2\pi iT}\int_{-i\infty}^{i\infty}\frac{1-G(s)}{s}\,\frac{1-G(-s)}{s}\,ds\right] + \frac{N}{2\pi i}\int_{-i\infty}^{i\infty}s^4G(s)G(-s)\,ds \tag{A3}$$

provided again that (29), (30), and the first of equations (38) hold.

The integrals occurring in equations (A1) and (A3) can be evaluated immediately by means of the tables E.2-1 in Appendix E of reference 12.

One further point which might be made is this. It is easy to show by use of equations (A2) where the parameter $(\overline{a_M^2}-\overline{a^2})/N\lambda^5$ came from. Notice that in equation (43) that $G(s)$ is really a function of s/λ, so that $G(\lambda s)$ does not involve λ explicitly at all. Hence, writing $s = \lambda p$ in (A2) and abbreviating $\overline{a_M^2}(T)$ to $\overline{a_M^2}$, we obtain

$$\overline{a_M^2} = \overline{a^2} + \frac{N\lambda^5}{2\pi i}\int_{-i\infty}^{i\infty}\left[p^4G(\lambda p)G(-\lambda p) - \frac{1-G(\lambda p)}{p}\,\frac{1-G(-\lambda p)}{p}\right]dp$$

or

$$\frac{\overline{a_M^2}-\overline{a^2}}{N\lambda^5} = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left[p^4G(\lambda p)G(-\lambda p) - \frac{1-G(\lambda p)}{p}\,\frac{1-G(-\lambda p)}{p}\right]dp$$

where the right-hand side is in a normalized form and, contrary to appearances, does not contain _any_ of the parameters $\overline{a^2}$, N, T, or λ.

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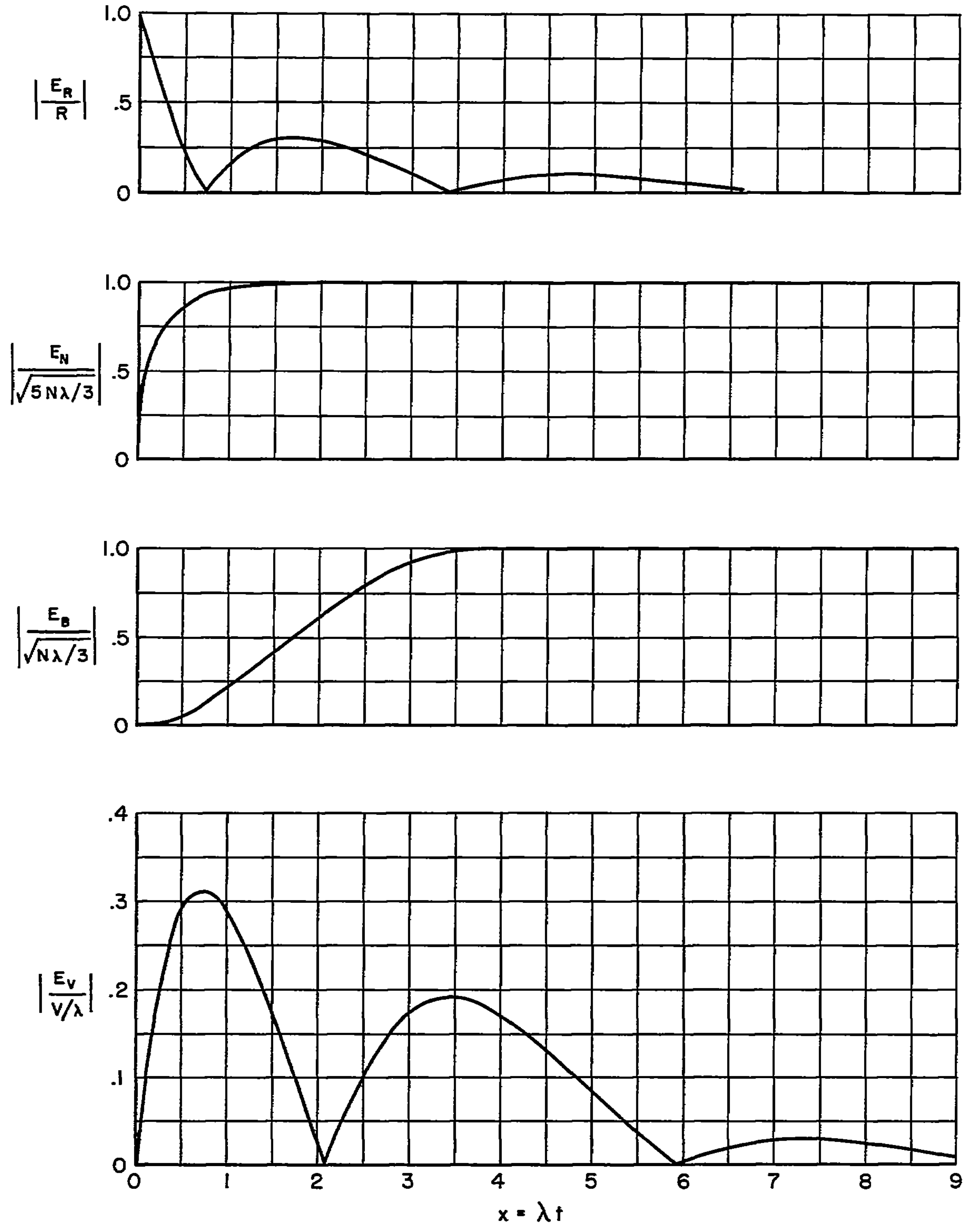

Figure 1.- The minimum error as a function of time, $E_{min} = \sqrt{E_R^2+E_N^2+E_V^2+E_B^2}$.

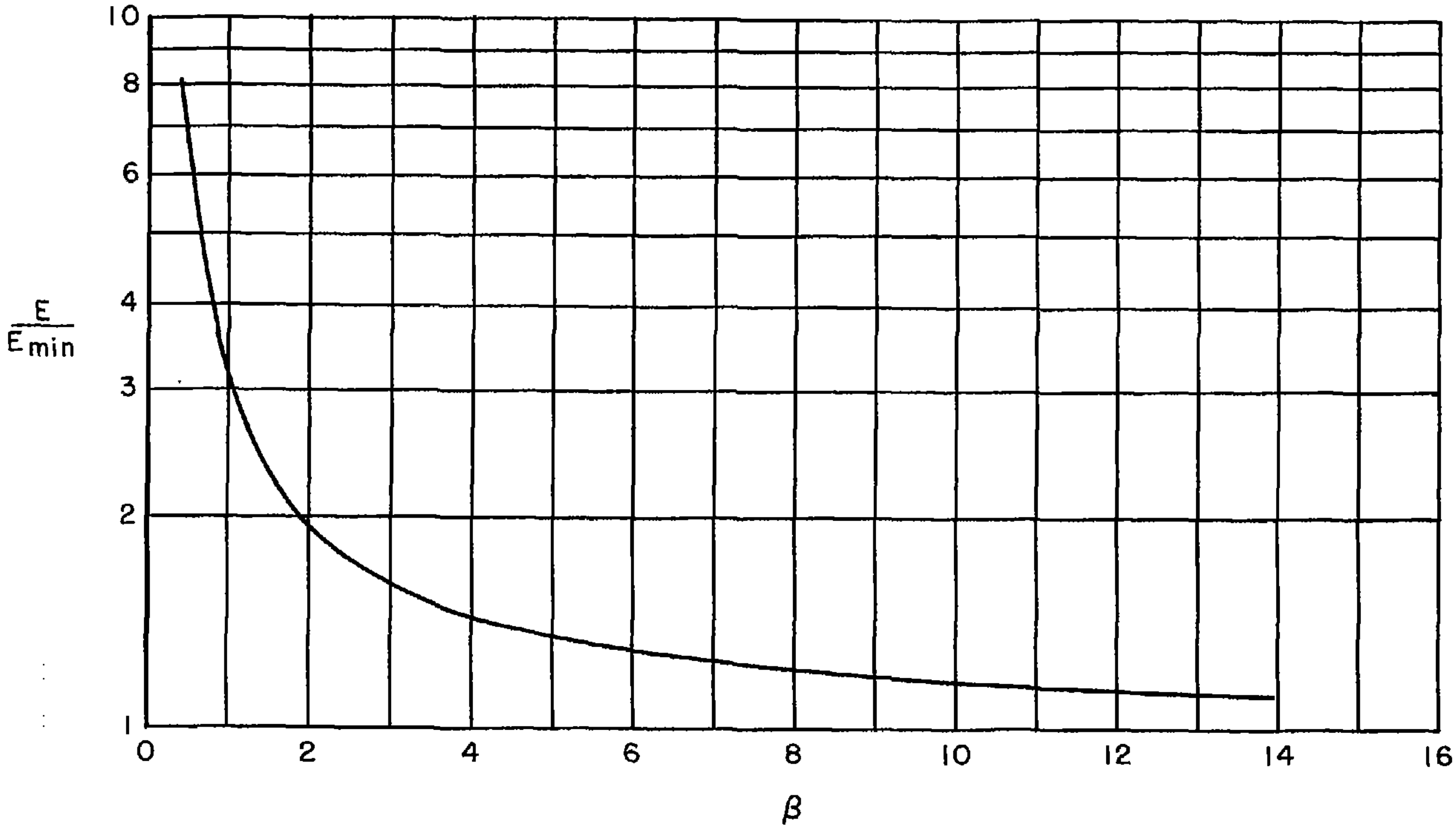

Figure 2.- Minimum error in the presence of acceleration limiting.

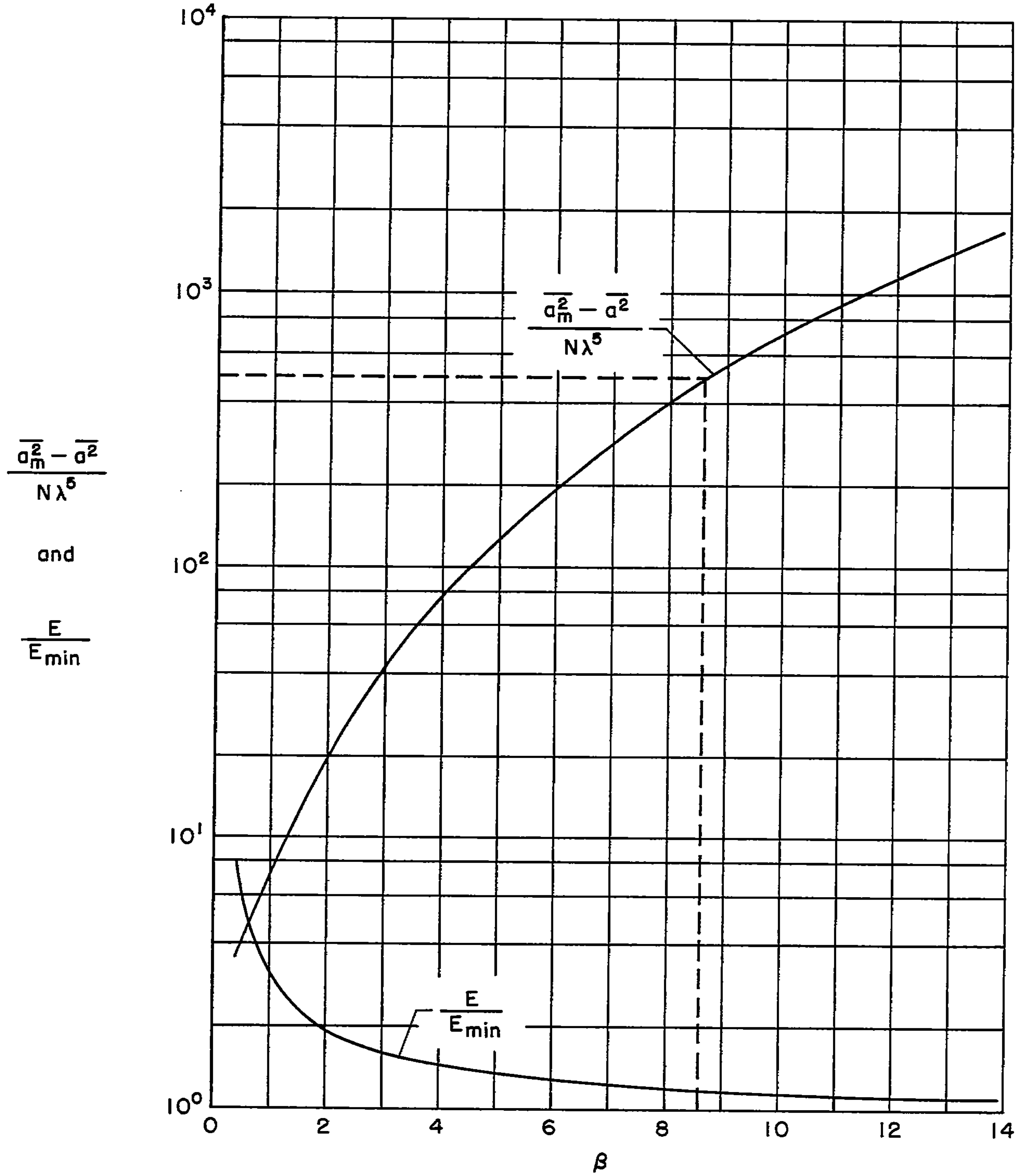

Figure 3.- Acceleration and error of optimum missile.

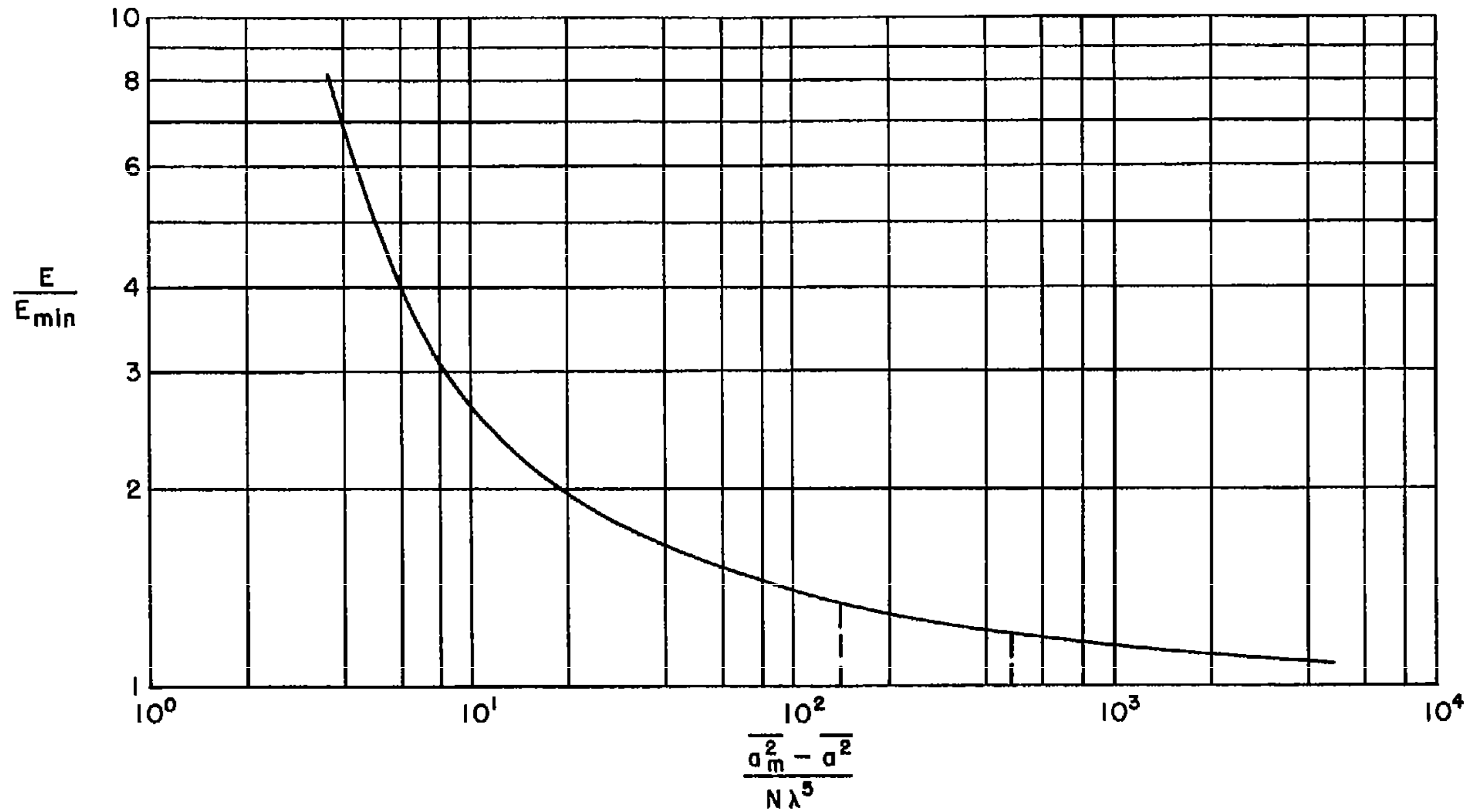

Figure 4.- Error versus dimensionless acceleration parameter.